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CONSISTENT AUTOREGRESSIVE SPECTRAL ESTIMATION FOR NOISE-CORRUPTED AUTOREGRESSIVE TIME SERIES

D.F. Gingras

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For the case when the observed series consists of the sum of an autoregressive process of known order and white noise the application of autoregressive spectral estimation methods may not be correct. The presence of the additive noise introduces zeros which are not adequately modeled by an autoregressive model. In this report an autoregressive spectral estimator for the noise-corrupted case is developed and shown to be consistent. The "high-order" Yule-Walker equations are used to estimate the autoregressive parameters from the noise-corrupted observations. A least squares estimate for the variance of the innovations sequence is also developed and shown to be consistent. These consistent estimates for the autoregressive parameters and the innovations variance are used to form the consistent autoregressive spectral estimates.

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SUMMARY

It has been shown that the application of AR spectral estimation methods to time series consisting of an AR process observed in white noise is not a correct application for the AR method. An autoregressive process plus noise (AR + N) has been shown to require an ARMA model for correct representation. The "higher-order" Yule-Walker equations were shown to provide consistent and unbiased estimates for the AR parameters of an AR + N series. A consistent estimate for the innovations variance of the unobservable AR process was also developed. The resulting spectral estimate for the AR series as observed in white noise was shown to be a consistent estimate. It now remains to be shown that the AR spectral estimate is unbiased and asymptotically normal. The structure of the asymptotic covariance matrix must also be evaluated.

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CONTENTS

I. INTRODUCTION

Parzen [1] and Burg [2], [3] proposed high resolution methods of power spectral estimation. Parzen's method is referred to as the autoregressive (AR) method and Burg's method as the maximum entropy method (MEM). The principal advantage of the AR and MEM methods over conventional smoothed periodogram methods is their enhanced resolution property. A secondary advantage, related to the enhanced resolution property, is that these methods do not require the use of a window function to achieve a consistent estimate. This results in both the resolution and stability being more directly related to the characteristics of the time series under observation. Van den Bos [4] showed that MEM spectral analysis is equivalent to fitting an AR model to the series thus establishing an equivalence between the two methods. Only the AR method will be discussed in this report.

Often the observed time series, Y_t, is not generated by a pure AR process but instead by the sum of an AR process and white noise. Let X_t be an AR process and, n_t, white noise, then the observed time series is given by

$$Y_t = X_t + n_t$$

and will be referred to as an AR + N process. The application of AR spectral estimation methods assumes that the observed time series can be fitted to a finite order AR process, that is, an all-pole spectral model. Pagano [5] showed that the correct model for an AR process plus noise is an autoregressive-moving average ARMA, pole-zero spectral model. Thus, when additive noise is present, the all-pole model assumption is violated, resulting in a degraded spectral estimate, Kay [6]. It has been shown experimentally that much of the enhanced resolution property is not realized when AR spectral estimation methods are applied to an AR process in noise, Marple [7]. The problem under investigation herein is the development of a consistent autoregressive spectral estimator that takes into account the presence of additive noise.

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[1] Parzen, E., "Statistical spectrum analysis (single channel case) in 1968." Stanford Univ. Tech. Report # 11 on Contract Nonr-225(80), 1968.

[2] Burg, J.P., "Maximum entropy spectral analysis," 37th Annual Int. Meeting Soc. of the Explor. Geophys., Oklahoma City, 1967.

[3] Burg, J.P., "A new analysis technique for time series data," Adv. Study Institute on Signal Processing, NATO, Enschede, Netherlands, 1968.

[4] van den Bos, A., "Alternative interpretation of maximum entropy spectral analysis," *IEEE Trans. Inform. Theory*, IT-17, 493-494, 1971.

[5] Pagano, M., "Estimation of models of autoregressive signal plus white noise." *Ann. Statist.*, vol 12, no. 1, 99-108, 1974.

[6] Kay, S.M., "The effects of noise on the autoregressive spectral estimator, *IEEE Trans. Acoust., Speech, Signal Processing*, vol ASSP-27, no. 5, 478-485, 1979.

[7] Marple, S.L., "Resolution of conventional Fourier, autoregressive and special ARMA methods of spectral analysis." Presented at IEEE Int. Conf. Acoustics, Speech and Signal Proc., Hartford, CT, 1977.

For autoregressive spectral estimation, without additive noise, methods exist for estimating the AR parameters and thus the spectrum. Asymptotic first- and second-order statistics, under varying conditions on the underlying process, have been derived by Akaike [8], Kromer [9], and Berk [10]. Overall the results indicate that, under proper conditions, the AR spectral estimates have asymptotic statistical properties similar to a smoothed periodogram with a truncated rectangular window function. Pagano [5] investigated the estimation of AR parameters in noise and showed that strongly consistent efficient estimates for the AR parameters could be obtained, but he did not address the problem of statistical properties for the spectral estimate. Kay [11] discusses a method that compensates for the presence of noise in AR spectral estimation, but his method requires knowledge of the noise variance. Also, he does not develop statistical properties for the AR parameter estimates or for the AR spectral estimates for his noise-compensated estimates.

In this report it is shown that the correct model for an AR + N series is a special form of the general ARMA model. In a previous report, Gingras [12], it was shown that when the "higher order" Yule-Walker (Y-W) equations are used to estimate the AR parameters for an AR + N series, the estimates are consistent and asymptotically normal. In this report it is shown that those consistent AR parameter estimates, along with a consistent least squares estimate for the variance of the AR innovations series form the basis for a consistent estimate of the AR process spectrum. The asymptotic statistics for the proposed spectral estimate are under investigation and will be presented in a future report.

In Section II the AR and ARMA processes and their power spectral densities are presented. Estimates for the AR parameters are discussed for both cases. It is shown, in Section III, that an autoregressive process plus white noise series, AR + N, is a special case of the more general ARMA series and that the "higher order" Y-W equations can be used to obtain a consistent estimate for the AR parameters. In Section IV, a spectral estimate is proposed for estimating the power spectrum of the AR series when the series is observed in additive noise. The proposed spectral estimate is shown to be a consistent estimate.

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8 Akaike, H., "Power spectrum estimation through autoregressive model fitting." *Ann. Inst. Statist. Math.*, vol 21, 407-419, 1969.

9 Kromer, R., "Asymptotic properties of the autoregressive spectral estimator" Ph.D. thesis, Dept. of Statistics, Stanford Univ., Stanford, CA, 1969.

10 Berk, K.N., "Consistent autoregressive spectral estimates." *Ann. Statist.*, vol 2, no. 3, 489-502, 1974.

11 Kay, S.M., "Noise compensation for autoregressive spectral estimates," *IEEE Trans. Acoust., Speech, Signal Processing,* vol ASSP-28, no. 3, 1980.

12 Gingras, D.F., "Asymptotic normality of autoregressive parameter estimates for mixed time series." Naval Ocean Systems Center, Tech. Report 733, 1981.

II. THE AUTOREGRESSIVE AND AUTOREGRESSIVE-MOVING AVERAGE PROCESSES

Assume we have observations $\left\{X_t\right\}_{t=1}^{N}$ of a stationary scalar zero mean discrete parameter time series generated by a finite order autoregressive (AR) process

$$X_t - a_1X_{t-1} - a_2X_{t-2} - \cdots - a_MX_{t-M} = \epsilon_t \tag{1}$$

where the innovations sequence of random variables $\left\{\epsilon_t\right\}$ is assumed to be independent normal identically distributed with zero mean and variance $\sigma_\epsilon{}^2$, i.e., i.i.d. $N(0, \sigma_\epsilon{}^2)$. To guarantee that the process $\left\{X_t\right\}$ is stationary, the roots of the polynomial

$$1 - a_1Z - \cdots - a_MZ^M$$

must lie outside the unit circle on the Z-plane.

The autocovariance function can be obtained by multiplying (1) by X_{t-k} and taking expectations term by term, we get

$$r_X(k) - a_1\, r_X(k-1) - \cdots - a_M\, r_X(k-M) = 0 \quad k > 0 \tag{2}$$

where $r_X(k) = E\left\{X_{t-k}\, X_t\right\}$. Note that the term $E\left\{X_{t-k}\, \epsilon_t\right\}$ does not exist for $k > 0$. This is the case since X_{t-k} can only be correlated with inputs ϵ_s for $s \leqslant t-k$. If we evaluate the autocovariance function (2) for $k = 1, 2, \cdots, M$ we obtain the Yule-Walker (Y-W) equations. The Y-W equations are a set of linear equations for the AR parameters in terms of the autocovariances r_X. In matrix form we have

$$\begin{bmatrix} r_X(0) & r_X(1) & \cdots & r_X(M-1) \\ r_X(-1) & r_X(0) & \cdots & r_X(M-2) \\ \vdots & \vdots & & \vdots \\ r_X(1-M) & r_X(2-M) & \cdots & r_X(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_M \end{bmatrix} = \begin{bmatrix} r_X(1) \\ r_X(2) \\ \vdots \\ r_X(M) \end{bmatrix} \tag{3}$$

where the M X M covariance matrix is a symmetric real Toeplitz matrix. When sample estimates for the covariances are substituted into the Y-W equations, the solution provides consistent estimates for the AR parameters. We can write the Y-W equations in terms of the estimates as

$$\sum_{j=1}^{M} \hat{a}_j\, \hat{r}_X(j-k) = \hat{r}_X(k) \qquad k = 1, 2, \cdots, M \tag{4}$$

where $\hat{r}_X(k) = \frac{1}{N} \sum_{t=1}^{N-|k|} X_t X_{t+|k|}$.

Mann and Wald [13], in a paper dealing with linear stochastic difference equations developed least squares estimates for the AR parameters and proved that the estimates were consistent and asymptotically unbiased. They proved that the parameter estimation errors were asymptotically normal and calculated the asymptotic error covariance matrix. Mann and Wald [13] also developed an estimator for the innovations process variance, σ_ϵ^2,

$$\hat{\sigma}_\epsilon^2 = \hat{r}_X(0) - \sum_{j=1}^{M} \hat{a}_j \hat{r}_X(j) \tag{5}$$

and showed it to be a consistent estimate.

The power spectral density for an AR process of finite order is written as

$$\phi_X(\lambda) = \frac{\sigma_\epsilon^2}{2\pi A(e^{i\lambda}) A(e^{-i\lambda})} \tag{6}$$

where $A(e^{i\lambda}) = 1 - \sum_{j=1}^{M} a_j \exp(ij\lambda)$. An estimator for the power spectrum of an AR time series that is often used is (6), with estimates for σ_ϵ^2 and the AR parameters used in place of the true quantities. Akaike [8], Kromer [9], and Berk [10] all have studied the asymptotic properties of this spectral estimator; their results are discussed in Section IV.

In Section III it is shown that the sum of an AR process of order (M) and noise can be modeled as an ARMA process of order (M, M). This process can be represented by

$$Y_t - a_1 Y_{t-1} - \cdots - a_M Y_{t-M} = \eta_t - b_1 \eta_{t-1} - \cdots - b_M \eta_{t-M} \tag{7}$$

where the sequence $\left\{\eta_t\right\}$ is assumed to be independent normal identically distributed with zero mean and variance σ_η^2, i.e., i.i.d. $N(0, \sigma_\eta^2)$.

We can evaluate the autocovariance function for this process by multiplying through by Y_{t-k}, and taking expectations term by term, we get

$$\begin{aligned} &E\left\{Y_{t-k} Y_t\right\} - a_1 E\left\{Y_{t-k} Y_{t-1}\right\} - \cdots - a_M E\left\{Y_{t-k} Y_{t-M}\right\} \\ &\quad = E\left\{Y_{t-k} \eta_t\right\} - b_1 E\left\{Y_{t-k} \eta_{t-1}\right\} - \cdots - b_M E\left\{Y_{t-k} \eta_{t-M}\right\} . \end{aligned} \tag{8}$$

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[13] Mann, H.B. and Wald, A., "On the statistical treatment of linear stochastic difference equations." *Econometrica*, vol II, 173-200, 1943.

Since Y_{t-k} depends only on inputs η_{t-j} for $t-j \leqslant t-k$ it follows that

$$E\left\{Y_{t-k}\,\eta_{t-j}\right\} = \begin{cases} 0 & k > j \\ r_{Y\eta}(j-k) & k \leqslant j\ . \end{cases}$$

In (8) we see that the range of j is 0 to M, thus if we define $r_Y(k) = E\left\{Y_{t-k}\,Y_t\right\}$ then we can write

$$r_Y(k) - a_1\, r_Y(k-1) - \cdots - a_M\, r_Y(k-M) = 0 \quad k = M+1, \cdots, 2M \tag{9}$$

or in matrix form, we have

$$\begin{bmatrix} r_Y(M) & r_Y(M-1) & \cdots & r_Y(1) \\ r_Y(M+1) & r_Y(M) & \cdots & r_Y(2) \\ \vdots & \vdots & & \vdots \\ r_Y(2M-1) & r_Y(2M-2) & \cdots & r_Y(M) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_M \end{bmatrix} = \begin{bmatrix} r_Y(M+1) \\ r_Y(M+2) \\ \vdots \\ r_Y(2M) \end{bmatrix} \tag{10}$$

These relationships are referred to as the "higher order" Yule-Walker equations. When sample estimates of the covariances are substituted into these Y-W equations the solution provides estimates for the AR parameters for a mixed autoregressive-moving average time series.

Lemma 1:

Given $\left\{Y_t\right\}$, observations of a finite order ARMA process, then estimates of the AR parameters $\left\{a_j\right\}_{j=1}^{M}$, estimated using the "higher order" Y-W equations, are consistent estimates.

Proof:

This result was proven by Gersch [14].

The problem of estimating the autoregressive parameters for a mixed autoregressive moving-average time series, when the process order is known, was first examined by Gersch [14]. Gersch [14] showed that when the "higher order" Y-W equations are used to estimate the AR parameters the estimates are asymptotically unbiased. The structure of the asymptotic error covariance matrix was also evaluated. More recently, Gingras [12] proved that when the "higher order" Y-W equations are used the AR parameter estimates are asymptotically jointly multivariate normal with zero mean and finite covariance matrix.

[14] Gersch, W., "Estimation of the autoregressive parameters of a mixed autoregressive moving average time series." *IEEE Trans. Automat. Contr.*, vol AC-15, 583-588, 1970.

The asymptotic error covariance matrix was evaluated and was shown to be equivalent to the result of Gersch [14].

For the mixed autoregressive-moving average case various methods exist for estimating the moving average (MA) parameters. For example, Box and Jenkins [15] provide a method for iteratively calculating estimates for the MA parameters from the AR parameter estimates and the sample covariance estimates. Similiarly, Anderson [16] and Hannan [17], [18] also provide methods for estimating the MA parameters of a mixed time series.

The power spectral density for an ARMA process is given by

$$\phi_Y(\lambda) = \frac{\sigma_\eta^{\,2}}{2\pi} \; \frac{B(e^{i\lambda})\; B(e^{-i\lambda})}{A(e^{i\lambda})\; A(e^{-i\lambda})}$$

where $A(e^{i\lambda})$ is as previously defined and $B(e^{i\lambda}) = 1 - \sum_{j=1}^{M} b_j \exp(ij\lambda)$. The MA parameters $\left\{ b_1, b_2, \cdots, b_M \right\}$ determine the zero locations and the AR parameter $\left\{ a_1, a_2, \cdots, a_M \right\}$ determine the pole locations. The composite response of the poles and zeros determine the spectral density for the ARMA process.

[15] Box, G.E.P. and Jenkins, G.M., *Time Series Analysis: Forecasting and Control,* San Francisco: Holden-Day, 1976.

[16] Anderson, T.W., *The Statistical Analysis of Time Series,* New York: John Wiley and Sons, Inc., 1971.

[17] Hannan, E.J., "The estimation of mixed moving average autoregressive systems." *Biometrika,* vol 56, 579-593, 1969.

[18] Hannan, E.J., *Multiple Time Series,* New York: John Wiley and Sons, Inc, 1970.

III. THE AUTOREGRESSIVE PROCESS IN NOISE (AR + N)

Assume that the observed discrete parameter time series is generated by the sum of a stationary autoregressive process $\{X_t\}$ of known order, M, given by

$$X_t - \sum_{j=1}^{M} a_j X_{t-j} = \epsilon_t \tag{11}$$

and a noise sequence $\{n_t\}$. As before, assume that the innovations sequence, $\{\epsilon_t\}$ is i.i.d. $N(0, \sigma_\epsilon^2)$, that the noise sequence $\{n_t\}$ is i.i.d. $N(0, \sigma_n^2)$ and that the series $\{X_t\}$ and $\{n_t\}$ are independent. Define the observed series $\{Y_t\}$ by

$$Y_t = X_t + n_t \tag{12}$$

where $\{X_t\}$ and $\{n_t\}$ are as defined above. Also define the series $\{v_t\}$ by

$$v_t = Y_t - \sum_{j=1}^{M} a_j Y_{t-j} \; . \tag{13}$$

The following two Lemma's are results developed by Pagano [5].

<u>Lemma 2:</u>

There exists a sequence of random variables $\{\eta_t\}$ that are i.i.d. $N(0, \sigma_\eta^2)$, defined on the same sample space as $\{Y_t\}$ and constants $\{b_j\}_{j=1}^{M}$ such that

$$v_t = \eta_t - \sum_{j=1}^{M} b_j \eta_{t-j} \tag{14}$$

where $\{v_t\}$ is a moving average series of order M.

Proof:

From (12) and (13) we have

$$v_t = -\sum_{j=0}^{M} a_j (X_{t-j} + n_{t-j}) \qquad a_0 = -1$$

$$= \epsilon_t - \sum_{j=0}^{M} a_j n_{t-j} \quad . \tag{15}$$

Since $\{v_t\}$ is stationary, we define

$$r_v(u) = E\{v_t v_{t+u}\}$$

where from (13) we see that $r_v(u)$ is zero for $|u| > M$. Therefore $\{v_t\}$ is a moving average series of order less than or equal to M. From (14) and (15) we have

$$r_v(M) = \sigma_\eta^{2} b_M = \sigma_n^{2} a_M \; .$$

Since, by definition, a_M is non-zero, then the parameter b_M must also be non-zero. Thus $\{v_t\}$ must be a moving average series of order M.

□

Lemma 3:

The series $\{Y_t\}$, as defined above, is a mixed autoregressive moving average series of order (M, M).

Proof:

Equating (13) and (14), taking Z transforms, we have

$$Y(Z)\, A(Z) = \eta(Z)\, B(Z).$$

From this relationship, with $Z = e^{i\lambda}$, it follows directly that the power spectral density of the process $\{Y_t\}$ is given by

$$\phi_Y(\lambda) = \frac{\sigma_\eta^2}{2\pi} \frac{B(e^{i\lambda})\,B(e^{-i\lambda})}{A(e^{i\lambda})\,A(e^{-i\lambda})} \;. \tag{16}$$

We can also express the power spectral density of $\left\{ Y_t \right\}$ as

$$\phi_Y(\lambda) = \frac{\sigma_n^2}{2\pi} + \frac{\sigma_\epsilon^2}{2\pi\, A(e^{i\lambda})\,A(e^{-i\lambda})} \;. \tag{17}$$

Thus we can write

$$\sigma_\eta^2\, B(Z)\, B(Z^{-1}) = \sigma_\epsilon^2 + \sigma_n^2\, A(Z)\, A(Z^{-1}) \;.$$

Therefore if Z_o is a root of $A(Z)$ then neither Z_o nor Z_o^{-1} is a root of $B(Z)$. Thus, $\phi_Y(\lambda)$ is the modulus squared of the ratio of two Mth degree polynomials with no common roots, therefore $\left\{ Y_t \right\}$ is a mixed autoregressive-moving average series of order (M, M).

□

As a result of Lemma 3 we can express the series $\left\{ Y_t \right\}$ as

$$Y_t - a_1 Y_{t-1} - \cdots - a_M Y_{t-M} = \eta_t - b_1 \eta_{t-1} - \cdots - b_M \eta_{t-M} \;.$$

Note that since the AR + N series can be expressed as an ARMA series the "higher order" Y-W equations, discussed in the previous section, can be used to estimate the AR parameters for this case.

Equation (17) expresses the power spectrum for the $\left\{ Y_t \right\}$ process in terms of its components, i.e., the spectrum for the noise and the spectrum for the AR process. We can rewrite (17) as

$$\phi_Y(\lambda) = \frac{\sigma_\epsilon^2 + \sigma_n^2\, A(e^{i\lambda})\,A(e^{-i\lambda})}{2\pi\, A(e^{i\lambda})\,A(e^{-i\lambda})} \tag{18}$$

This form of the spectral density provides insight into the characteristics of the AR + N process. We see from (18) that the spectral density includes both poles and zeros. We also note that the zeros have been introduced by the presence of the additive noise and that the zero locations are related to the pole locations and variances σ_ϵ^2, σ_n^2. In Kay [6] the roots of

$$\sigma_\epsilon^2 + \sigma_n^2\, A(Z)\, A(Z^{-1})$$

are examined as a function of signal-to-noise ratio. It was shown that for a high signal-to-noise ratio case (i.e., $\sigma_\epsilon{}^2 \gg \sigma_n{}^2$) the zeros are located near the origin on the Z-plane. As the signal-to-noise ratio decreases, the zeros move toward the pole locations, cancelling the effect of the poles, producing a flat noise-like spectral estimate.

IV. A CONSISTENT AR SPECTRAL ESTIMATE

When the observed time series $\{X_t\}$ can be fitted to a finite order AR model the following AR spectral estimator is often used

$$\hat{\phi}_X(\lambda) = \frac{\hat{\sigma}_\epsilon^2}{2\pi\, \hat{A}(e^{i\lambda})\, \hat{A}(e^{-i\lambda})}$$

where $\hat{A}(e^{i\lambda}) = 1 - \sum_{k=1}^{M} \hat{a}_k \exp(ik\lambda)$ and the set $\left\{\hat{a}_k\right\}_{k=1}^{M}$ are the estimated AR parameters. The estimate for σ_ϵ^2, as defined by (5), is a function of the estimated AR parameters and the sample covariances. Akaike [8], Kromer [9], and Berk [10] have studied the asymptotic statistical properties of this spectral estimate when the AR parameters are estimated via the Y-W equations. Akaike [8], under the assumption that the process order M is finite and known, proved that the AR spectral estimate is asymptotically normal and evaluated the asymptotic covariance matrix for the limit distribution as $N \to \infty$. Kromer [9], under assumptions relative to the smoothness and boundedness of the spectral density, proved that the AR spectral estimates are consistent and asymptotically normal as first $N \to \infty$ then $M \to \infty$. Kromer [9] also evaluated the covariance matrix of the limit distribution and provided an indication of the rate of decrease for the bias. Berk [10], under assumptions similar to Kromer's and with assumptions on the asymptotic rates for N and M proved that the AR spectral estimates are consistent, asymptotically normal and uncorrelated at different frequencies. Berk [10] and Kromer [9] both proved that the asymptotic variance for the spectral estimate is the same as that for periodogram estimate based on a rectangular window truncated at $\pm M$.

When the observed time series $\{Y_t\}$ has been generated by an AR process plus white noise, we showed in Section III that this series can be modeled as an ARMA process with power spectral density given by

$$\phi_Y(\lambda) = \frac{\sigma_\eta^2}{2\pi} \frac{B(e^{i\lambda})\, B(e^{-i\lambda})}{A(e^{i\lambda})\, A(e^{-i\lambda})} . \tag{19}$$

Methods are available for estimating the AR and MA parameters for an ARMA series. Asymptotically efficient estimates have been developed, usually through the application of iterative solutions to nonlinear schemes (see Parzen [19], Hannan [17], [18], and Anderson [16]). Recently, Hannan [20] developed a recursive method for estimating the parameters for an

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[19] Parzen, E., "Efficient estimation of stationary time series mixed schemes," Stanford Univ. Tech. Report #16 on Contract Nonr-225(80), 1971.

[20] Hannan, E.J., "Recursive estimation based on ARMA models," *Ann. Statist.*, vol 8, no. 4, 762-777, 1980.

ARMA series. With estimates for σ_η^2, $B(e^{i\lambda})$ and $A(e^{i\lambda})$, one could construct an estimate for $\phi_Y(\lambda)$ using (19). The methods available for estimating the MA parameters are quite complex, and we note that the spectral estimate desired is that for $\phi_X(\lambda)$, not $\phi_Y(\lambda)$. Therefore, we see that if we could estimate σ_ϵ^2 and $A(e^{i\lambda})$ from observations of the noise-corrupted series $\{Y_t\}$ then we would have an estimate for the AR spectrum $\phi_X(\lambda)$.

Pagano [5] addressed the problem of estimating the parameters σ_n^2, σ_ϵ^2 and $\{a_k\}_{k=1}^{M}$ for an AR + N series. Under assumptions that the noise and the innovations series are normally distributed through the application of nonlinear regression methods he proved that efficient estimates of the parameters can be constructed. Pagano [5] did not address the estimation of the spectral density.

The objective of the work herein was the development of a consistent spectral estimate for the AR time series when the observations are corrupted by additive white noise. It was previously shown, in Lemma 1, that consistent estimates for the AR parameters can be constructed through the use of the "higher order" Y-W equations. Thus, it remains to develop a consistent estimate for the variance of the innovations series, σ_ϵ^2, from observations of the noise-corrupted series.

It was shown in Section III that the power spectral density for the AR + N process can be written as

$$\phi_Y(\lambda) = \frac{\sigma_n^2}{2\pi} + \frac{\sigma_\epsilon^2}{2\pi\, A(e^{i\lambda})\, A(e^{-i\lambda})} \,. \tag{20}$$

We now use the linear relation of (20) in conjunction with the AR parameter estimates $\{\hat{a}_k\}_{k=1}^{M}$ and a smoothed periodogram estimate for $\phi_Y(\lambda)$ to develop an estimate for the variance of unobservable innovations series, ϵ_t. Let $\hat{J}_N(\lambda)$ be the smoothed periodogram estimate for Y_t, we have

$$\hat{J}_N(\lambda) = \frac{1}{2\pi} \sum_{k=-(N-1)}^{N-1} h(k\,B_N)\ \hat{r}_Y(k) \exp(-ik\lambda) \tag{21}$$

where $\hat{r}_Y(k)$ is the covariance estimate given by

$$\hat{r}_Y(k) = \frac{1}{N} \sum_{t=1}^{N-|k|} Y_t\, Y_{t+|k|}$$

and $h(k B_N)$ is the covariance averaging kernel defined as the Fourier transform of the weight function defined by Brillinger [21]. The sequence B_N, N = 1, 2, ⋯ is a sequence of scale parameters with the properties $B_N > 0$, $B_N \to 0$ and $B_N N \to \infty$ as $N \to \infty$.

From (20), for all $\lambda_j = \frac{j\pi}{M-1}$, j = 0, 1, ⋯ , M−1, we can write the linear relation

$$2\pi \hat{J}_N(\lambda_j) = \sigma_n^2 + \frac{\sigma_\epsilon^2}{\hat{A}(e^{i\lambda_j}) \hat{A}(e^{-i\lambda_j})} + u_j \tag{22}$$

where u_j is an undefined random error term. We propose as estimators for σ_n^2 and σ_ϵ^2 the least squares estimates derived from (22). Define $\underline{\Phi}$ to be a (M X 1) vector of estimates $2\pi \hat{J}_N(\lambda_j)$ j = 0, 1, ⋯ , M−1, and let $\hat{g}(\lambda_j) = \hat{A}(e^{i\lambda_j}) \hat{A}(e^{-i\lambda_j})$. Define the (M X 2) matrix $\underline{H}$ by

$$\underline{H}^T = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \frac{1}{\hat{g}(\lambda_0)} & \frac{1}{\hat{g}(\lambda_1)} & \cdots & \frac{1}{\hat{g}(\lambda_{M-1})} \end{bmatrix} .$$

Define $\underline{\Theta}^T \triangleq \left[\sigma_n^2, \sigma_\epsilon^2\right]$ and $\underline{u}^T \triangleq \left[u_0, u_1, \cdots, u_{M-1}\right]$ then we can write the matrix equivalent of (22) as

$$\underline{\Phi} = \underline{H}\, \underline{\Theta} + \underline{u} .$$

The well known least squares estimate for $\underline{\Theta}$ is then given by the solution of the normal equation

$$(\underline{H}^T \underline{H})\, \hat{\underline{\Theta}} = \underline{H}^T \underline{\Phi} .$$

Define γ_M by

$$\gamma_M \triangleq M \sum_{j=0}^{M-1} \left(\frac{1}{\hat{g}(\lambda_j)}\right)^2 - \left(\sum_{j=0}^{M-1} \frac{1}{\hat{g}(\lambda_j)}\right)^2$$

then the solution of the normal equation yields the following relationships

$$\begin{bmatrix} \hat{\sigma}_n^2 \\ \hat{\sigma}_\epsilon^2 \end{bmatrix} = \frac{1}{\gamma_M} \begin{bmatrix} \sum_{j=0}^{M-1} \left(\frac{1}{\hat{g}(\lambda_j)}\right)^2 & -\sum_{j=0}^{M-1} \frac{1}{\hat{g}(\lambda_j)} \\ -\sum_{j=0}^{M-1} \frac{1}{\hat{g}(\lambda_j)} & M \end{bmatrix} \begin{bmatrix} 2\pi \sum_{j=0}^{M-1} \hat{J}_N(\lambda_j) \\ 2\pi \sum_{j=0}^{M-1} \frac{\hat{J}_N(\lambda_j)}{\hat{g}(\lambda_j)} \end{bmatrix}$$

[21] Brillinger, D.R., *Time Series Data Analysis and Theory*, New York: Holt, Rinehart and Winston, Inc., 1975.

It follows that the least squares estimate of the innovations series variance is given by

$$\hat{\sigma}_\epsilon^2 = \frac{2\pi M \sum_{j=0}^{M-1} \frac{\hat{I}_N(\lambda_j)}{\hat{g}(\lambda_j)} - 2\pi \sum_{j=0}^{M-1} \hat{I}_N(\lambda_j) \sum_{j=0}^{M-1} \frac{1}{\hat{g}(\lambda_j)}}{M \sum_{j=0}^{M-1} \left(\frac{1}{\hat{g}(\lambda_j)}\right)^2 - \left(\sum_{j=0}^{M-1} \frac{1}{\hat{g}(\lambda_j)}\right)^2} . \tag{23}$$

Lemma 4:

Given consistent estimates of the AR parameters $\left\{a_k\right\}_{k=1}^{M}$ then the least squares estimate for σ_ϵ^2, as given by (23), is a consistent estimate.

Proof:

We have from Lemma 1 that $\operatorname{plim}_{N\to\infty} \hat{a}_k = a_k$ for $k = 1, 2, \cdots, M$, thus it follows that

$$\hat{A}(e^{i\lambda_j}) \xrightarrow[N\to\infty]{P} A(e^{i\lambda_j})$$

for $j = 0, 1, \cdots, M-1$, and correspondingly

$$\hat{A}(e^{-i\lambda_j}) \xrightarrow[N\to\infty]{P} A(e^{-i\lambda_j}) .$$

Therefore, we can conclude that

$$\hat{g}(\lambda_j) \xrightarrow[N\to\infty]{P} g(\lambda_j)$$

for $j = 0, 1, \cdots, M-1$. Given the smoothed periodogram estimate, as previously defined, we know from Brillinger [21] that $\hat{I}_N(\lambda)$ converges in mean square to $\phi_Y(\lambda)$ uniform in λ, thus

$$\hat{I}_N(\lambda) \xrightarrow[N\to\infty]{P} \phi_Y(\lambda)$$

uniform in λ. From (23) we can write

$$\underset{N\to\infty}{\text{plim}}\ \hat{\sigma}_\epsilon^2 = \frac{\frac{2\pi}{M}\sum_{j=0}^{M-1}\frac{\phi(\lambda_j)}{g(\lambda_j)} - \frac{2\pi}{M^2}\sum_{j=0}^{M-1}\phi(\lambda_j)\sum_{j=0}^{M-1}\frac{1}{g(\lambda_j)}}{\frac{1}{M}\sum_{j=0}^{M-1}\left(\frac{1}{g(\lambda_j)}\right)^2 - \left(\frac{1}{M}\sum_{j=0}^{M-1}\frac{1}{g(\lambda_j)}\right)^2} \quad . \tag{24}$$

Using the relationship

$$2\pi\,\phi_Y(\lambda) = \sigma_n^2 + \sigma_\epsilon^2/g(\lambda)$$

the numerator of (24) becomes

$$\sigma_n^2/M \sum\frac{1}{g(\lambda_j)} + \sigma_\epsilon^2/M \sum\left(\frac{1}{g(\lambda_j)}\right)^2 - \frac{1}{M^2}\sum\frac{1}{g(\lambda_j)}\sum\left(\sigma_n^2 + \frac{\sigma_\epsilon^2}{g(\lambda_j)}\right)$$

$$= \sigma_n^2/M \sum\frac{1}{g(\lambda_j)} - \sigma_\epsilon^2/M \left(\frac{1}{g(\lambda_j)}\right)^2 - \sigma_n^2/M \sum\frac{1}{g(\lambda_j)} - \sigma_\epsilon^2/M^2\left(\sum\frac{1}{g(\lambda_j)}\right)^2$$

$$= \sigma_\epsilon^2\left\{\frac{1}{M}\sum\left(\frac{1}{g(\lambda_j)}\right)^2 - \left(\frac{1}{M}\sum\frac{1}{g(\lambda_j)}\right)^2\right\} \quad . \tag{25}$$

Substituting (25) for the numerator of (24) we have

$$\underset{N\to\infty}{\text{plim}}\ \hat{\sigma}_\epsilon^2 = \sigma_\epsilon^2$$ □

Lemma 5:

Given consistent estimates for the AR parameters $\left\{a_k\right\}_{k=1}^{M}$ and the consistent estimate for σ_ϵ^2 then the spectral estimate for the unobservable series $\left\{X_t\right\}$ given by

$$\hat{\phi}_X(\lambda) = \frac{\hat{\sigma}_\epsilon^2}{2\pi\,\hat{A}(e^{i\lambda})\,\hat{A}(e^{-i\lambda})}$$

is a consistent estimate.

Proof:

The result follows directly from the consistency of the estimated AR parameters and the estimate for σ_ϵ^2.

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